THE PRESSURE OF SOUND.

By Warren Weaver.

Synopsis.

The Pressure of Sound: Relation between Pressure and Energy Density.—An argument is given, following a method used by Larmor, to show that a certain general type of radiation will exert a pressure. The pressure of small sound waves is found to agree with this result, but for finite waves the conditions for the application of the argument are not satisfied. These finite waves do exert a pressure which depends upon the relation between pressure and density, the pressure being zero in a certain important case. This theory has been developed by Lord Rayleigh. It however appears that any actual aërial wave does exert a pressure not zero. The pressure on an absorbing sphere is a second order effect in the product (ak), and is therefore not considered in the usual treatment of spherical obstacles. Waves of energy density of 0.5 ergs/cm.^{3} or greater apparently must be treated as finite.

1. It is interesting and surprising that the subject of the steady pressure of sound waves on a surface normal to the direction of propagation has been so little mentioned in the ordinary literature of the subject. One finds, for example, no mention of it in Rayleigh's treatise, in Lamb's Dynamical Theory of Sound, and in many standard texts on physics. It is treated in two articles by Rayleigh,[1] in one by Poynting,[2] and in the article on sound by Stokes in the Encyclopedia Britannica. There is an apparent but entirely superficial confusion in the treatments here cited which it is the purpose of this note to remove.

2. To Larmor is due a method of argument to show that any propagated disturbance in which the energy density in the beam is inversely proportional to the square of the wave-length will exert a radiation pressure. For let the disturbance be propagated with a velocity c, and let it be reflected by a plane normal to the direction of propagation moving with a velocity of magnitude v opposite to c. Then by Doppler's Principle the wave-length of the reflected beam will be reduced in the ratio $1 - 2v/c$ to 1, so that the energy density in the reflected beam will be increased in the ratio $1 + 4v/c$ to 1, (v/c being supposed small). Let e be the energy density in the original beam, and consider unit area of the reflecting surface. An amount of energy $e(c + v)$ will be encountered by it per second. It will be reflected in a wave train that is shorter than $(c + v)$ in the ratio $1 - 2v/c$ to 1, but in which the energy density is

[1] 1905, II., p. 364; 1902, I., p. 338, Phil. Mag.
[2] 1905, I., p. 393, Phil. Mag.

larger in the ratio $1 + 4v/c$ to 1. Accordingly there will be added, per second, energy equal to

$$e(c + v)(1 - 2v/c)(1 + 4v/c) - e(c + v) = e(c + v)2v/c.$$

This energy is supplied by work done in advancing the reflecting surface a distance v per second against a pressure p. So that:

$$pv = e(c + v)2v/c$$

or

$$p = e(c + v)2/c.$$

The total energy density is the energy density in the oncoming and in the reflected beams, so that, if we denote it by E,

$$E = e + e(1 + 4v/c) = 2e(1 + 2v/c),$$

and if we set

$$p = \kappa E$$

we have, neglecting first order terms in v/c,

$$\kappa = 1$$

or

$$p = E, \tag{1}$$

so that the pressure is equal to the total energy density in the wave train if the reflector is moving slowly as compared to the velocity of propagation of the disturbance. In case the reflector is not moving at all, so that $v = 0$ the result (1) is rigidly correct. This argument of course covers the case of light pressure on a normal reflecting surface.

The argument, as given by Larmor,[1] is not restricted to first order terms in v/c. An incident train of length $c + v$ is reflected into a train of length $c - v$. The energy density in the reflected train is accordingly

$$e\left(\frac{c + v}{c - v}\right)^2$$

and the total energy reflected per second

$$e\left(\frac{c + v}{c - v}\right)^2 (c - v).$$

We have then for the increase in energy per second

$$e\left(\frac{c + v}{c - v}\right)^2 (c - v) - e(c + v) = e(c + v)\left[\frac{c + v}{c - v} - 1\right] = pv.$$

The total energy density in front of the reflector is

$$E = e + e\left(\frac{c + v}{c - v}\right)^2$$

[1] Art. on Radiation, Enc. Brit., 11th ed.

and if we set as before

$$p = \kappa E$$

or

$$\frac{e(c+v)}{v}\left[\frac{c+v}{c-v} - 1\right] = \kappa e\left[1 + \left(\frac{c+v}{c-v}\right)^2\right]$$

we have

$$\kappa = \frac{c^2 - v^2}{c^2 + v^2}, \tag{2}$$

which reduces to unity as in (1) if we neglect second order terms in v/c.

3. Let us consider a simple harmonic train of waves travelling in the positive x-direction given by

$$\varphi = Ae^{i(nt-kx)}. \tag{3}$$

The energy per unit volume of this plane wave disturbance is inversely proportional to the square of the wave-length, and hence sound of this sort should cause a pressure upon a reflecting surface equal to the total energy density in the incident and reflected sound beams according to 2. We shall see, in fact, from elementary mechanical principles that in the case of a perfectly absorbing plane obstacle normal to x there is also a pressure equal to the total energy density in front of the obstacle. Let R be the absorbing surface, and let a, b, c, d be a column one square centimeter in cross section, of any length, and normal to R. Since in

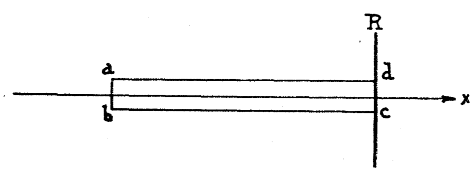

Fig. 1.

a steady state the air within the column neither gains nor loses momentum the momentum flow across a–b per second will be equal to the pressure on R. On the average there will be no momentum flow due to the variable part of the pressure, that is due to $\rho(\partial\varphi/\partial t)$, since this obviously has a time average of zero. The steady pressure at a–b will of course cause a steady pressure on R, but with that we are not concerned. The volume of air, however, gains forward momentum by having air enter it moving to the right, and as well by having air leave it moving to the left. If the velocity of the air particles be u the instantaneous rate at which momentum is flowing to the right across the surface a–b is then ρu^2: so that

$$p = \rho u^2 = \rho\left[\frac{\partial\varphi}{\partial x}\right]^2.$$

But from (3) we have

$$\frac{\partial \varphi}{\partial x} = - ikAe^{i(nt-kx)},$$

the real part of which

$$= kA \sin (nt - kx).$$

We have, therefore,

$$p = \rho k^2 A^2 \sin^2 (nt - kx),$$

the average value of which over a whole number of periods

$$= \tfrac{1}{2}\rho k^2 A^2, \tag{4}$$

the known expression for the energy density in the incident beam. Equation (4) also gives, as is well known, the value approached by the average over any lapse of time as the interval becomes long with respect to the period.

4. Expression (3) above is the velocity potential for plane waves under the assumption of small particle velocities, or, what comes to the same thing, under assumption that the pressure variation is so small that the volume modulus of elasticity of the gas may be considered constant over this pressure range. The relation between pressure and density in an actual gas is however such that a wave cannot be propagated without change of type (Rayleigh, Theory of Sound, Art. 250). An approximate study of this change of type shows that the pressure crests travel with higher velocity than do the pressure troughs, so that there is a tendency for the wave to "comb over" as a water wave does near the beach. This implies that the forward flow of momentum across a surface will be larger than in the case just considered, since the more dense portions of the gas are moving forward the more rapidly. There is actually a resultant forward flow of matter, which a reflecting or absorbing surface would have to reverse or annihilate, so that we should expect the pressure upon it to be greater than in the case given by (3). This problem has been treated by Rayleigh and Lamb. Rayleigh abandons the ordinary sound equations and starts from the basis of Bernoulli's equation. Lamb retains the ordinary equations but finds a corrective term to the expression for the change in pressure due to a small change in volume, the correction being obtained from the ordinary gas law. The result obtained is that the pressure is given by the equation

$$p = \tfrac{1}{2}(\gamma + 1)(\text{average energy/cubic centimeter}). \tag{5}$$

This is, in the first place, a hybrid result. As actually obtained the pressure is given by

$$p = \tfrac{1}{2}(\gamma + 1)\frac{1}{c}\int_0^c \rho_0 u^2 dx.$$

The expression

$$\frac{1}{c}\int_0^c \rho_0 u^2 dx,$$

is the value of the average energy density under the assumption of small pressure changes, while the coefficient $\frac{1}{2}(\gamma + 1)$ differs from unity only because the pressure changes have not been assumed to be so small. The result is therefore not to be considered as establishing an exception to the principle given in 2. In fact in this case we could not apply the principle given by Larmor at all because, there being continual change of type, there is, strictly speaking, nothing one can call the wave-length at all.

5. If the changes of pressure are small enough a wave can be propagated without change of type whatever the law between pressure and density. This is equivalent to saying that whatever the relation between p and ρ a sufficiently short piece of the p, v curve may be considered a straight line, such a relation being that which makes possible propagation without change of type.[1] Within the range over which this approximation is allowable the pressure will be numerically equal to the energy density in the sound-filled space before the reflector or absorber. In the case of finite waves, as suggested above, the pressure is in general larger than this. The exact relationship depends upon the law connecting pressure with density. It is given by equation (5) when the law of pressure is that given by the adiabatic relationship

$$p/p_0 = (\rho/\rho_0)\gamma.$$

(5) reduces to (1) in case we have Boyle's law. For the case of the law

$$p = \text{const} - d^2\rho_0^2/\rho, \tag{6}$$

which is the only relation under which there can be propagation without change of type, the pressure comes out curiously enough to be zero. Lord Rayleigh therefore remarks that "pressure and momentum are here associated with the tendency of waves to alter their forms as they proceed on their course." This might seem to imply that waves whose type is preserved as they move do not exert a pressure and have no momentum associated with them. In the case of actual waves, however, equation (6) holds only over pressure ranges so small that (3) accurately gives the velocity potential, and (4) gives the pressure. For these very small waves Lord Rayleigh has found a pressure equivalent to that given by (4).[2] The conclusion is, then, that any actual aerial waves, whether

[1] Lamb, Dynamical The. of Sound, p. 175.

[2] Phil. Mag., 1902, p. 338.

of such magnitude as to be considered small or finite, whether their type is preserved or not, do exert a pressure.

Both equations (1) and (5) have been made the basis of experimental determinations of the energy of sound waves. W. Zernov[1] used powerful waves of frequency 512 and energy density of the order of 0.5 ergs/cm^3., and found that equation (5) gave results which checked with a maximum discrepancy of 3 per cent. those given by a vibration-manometer method as developed by Wien.[2] W. Altberg's measurements[3] were on sound waves whose energy content was about half the above value, and he used equation 1. (His experiments were made before the publishing of Lord Rayleigh's 1905 paper.) It is unfortunately not possible to deduce from his results whether (1) or (5) represents the more closely the truth for sounds of this intensity since he considered only the constancy of the ratio of the result obtained by the pressure method to that given by the vibration-manometer method. This ratio was found to be approximately constant, as would of course be the case whether (1) or (5) was used.

6. The ordinary theory of the impinging of plane waves of sound on an obstructing sphere is upon the basis that (ka) is small, where a is the radius of the sphere. Since it exerts a pressure, we may associate a momentum with a sound beam, and since a perfectly absorbing sphere would annihilate per second the sound contained in a cylinder c in length and πa^2 in cross-section, it would be subjected to a pressure equal to $\frac{1}{2}\pi\rho A^2(ka)^2$. It is thus clear that for obstacles small enough to have the ordinary theory apply to them with good approximation the pressure effect we are considering would be negligible, containing as it does the square of (ka). This explains the absence of reference to any such pressure in the ordinary treatments. For sound in air of frequency 1,000 per second the product (ka) is equal to 0.01 (whose square we might perhaps agree to neglect) when $a = 0.0525$. Obviously, however, a criterion for how small particles could be and have the steady pressure effect sensible would have to take account also of density and amplitude. In certain experiments carried out during the war use was made of a small absorbing cylinder to measure pressure and hence energy density of super-sonic waves in water. The wave-lengths used were about those of the upper limit of audibility in air, their frequency being about four times this limit. The product (ka) in these experiments was approximately unity, so that the ordinary theory is entirely inapplicable, while the pressure is sensible and can be easily measured.

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[1] Ann. d. Phy., 21, 1906, p. 131.

[2] Wied. Ann., 36, 1889, p. 835.

[3] Ann. d. Phy., 11, 1903, p. 405.